Lecture 14

4. Hamiltonian Dynamics

Idea: Reformulate a dynamical system described by a Lagrangian $\mathcal{L}$ in terms of

$$q_i, \; p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}, \quad i \in \{1, \dots, N\}$$

solve this for $\dot{q}_i$ in terms of q_i, p_i

Start with the Lagrangian descriptions

- start with co-ordinates $q_1, \dots, q_N \equiv \underline{q}$
- set of conjugate momenta $p_1, \dots, p_N \equiv \underline{p}$ where

$$\boxed{p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}}$$

Step 1: Define $\dot{q}_i$ in terms of $\underline{p}, \underline{q}$ by solving relation for $\dot{q}_i$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad i \in \{1, \dots, N\}$$

Step 2: Define Hamiltonian

$$\boxed{H(\underline{q}, \underline{p}) = \sum_{i=1}^{N} \underbrace{\dot{q}_i}_{\dot{q}_i(\underline{q},\underline{p})} p_i - \mathcal{L}(\underline{q}, \underbrace{\dot{\underline{q}}}_{\dot{q}_i(\underline{q},\underline{p})})}$$

$= \dot{\underline{q}} \cdot \underline{p}$ (dot product)

(i) Compute $\frac{\partial H}{\partial q_j}$ $\quad j \in \{1, \dots, N\}$

chain rule

$$\frac{\partial H}{\partial q_j} = \sum_{i=1}^{N} \frac{\partial \dot{q}_i}{\partial q_j} p_i - \frac{\partial \mathcal{L}}{\partial q_j} - \sum_{K=1}^{N} \frac{\partial \dot{q}_K}{\partial q_j} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_K}}_{p_K}$$

$$= -\frac{\partial \mathcal{L}}{\partial q_j} = \frac{d}{dt} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{p_j} = -\dot{p}_j$$

and therefore

$$\boxed{\dot{p}_j = -\frac{\partial H}{\partial q_j}} \qquad (1)$$

(ii) Compute $\frac{\partial H}{\partial p_j}$

$$\text{So}\quad \frac{\partial H}{\partial p_j} = \sum_{i=1}^{N} \dot{q}_i \underbrace{\frac{\partial p_i}{\partial p_j}}_{\delta_{ij}} + \sum_{i=1}^{N} \frac{\partial \dot{q}_i}{\partial p_j} p_i - \sum_{k=1}^{N} \frac{\partial \dot{q}_k}{\partial p_j} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_k}}_{p_k}$$

chain rule

$$= \dot{q}_j$$

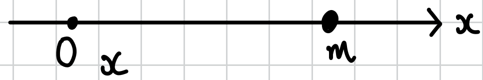
$$\Rightarrow \boxed{\dot{q}_j = \frac{\partial H}{\partial p_j}} \qquad (2)$$

(1) and (2) together are Hamilton's eqn for the system

- set of $2N$ first order equations for the time evolution of q, p a set of $2N$ variables
- set of p and q form the phase space for the system

4.2 Examples

(1) Particle of mass m on the x axis subjected to a force



$$F = -\frac{dV}{dx}$$

$$\text{So}\quad \mathcal{L}(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$$

The momentum conjugate to x is

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p_x}{m}$$

Then the Hamiltonian is

$$\dot{x} p_x - \mathcal{L} = \frac{p^2}{m} - \left(\frac{1}{2} m \frac{p^2}{m^2} - V\right)$$

$$\Rightarrow \boxed{H = \frac{p_x^2}{2m} + V}$$

Then Hamilton's equations are

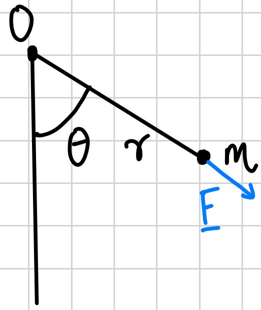
$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p}{m}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}$$

Newtons equation $\frac{dP}{dt} = -\frac{dV}{dx} \Rightarrow m\ddot{x} = -\frac{dV}{dx}$

(2) Particle on a plane

use polar co-ordinates subject to a force



$$\underline{F} = -\underline{\nabla}(V(r)) = -\frac{\underline{r}}{r} V'(r)$$

$$K.E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2), \quad PE = V(r), \quad \mathcal{L} = KE - PE$$

$$\Rightarrow \mathcal{L}(r, \theta, \dot{r}, \dot{\theta}) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$$

and $p_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$

Further,

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{mr^2}$$

The Hamiltonian is

$$H(r, \theta, \dot{r}, \dot{\theta}) = \dot{r}p_r + \dot{\theta}p_\theta - \mathcal{L}$$

$$= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \left(\frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - V\right)$$

$$\Rightarrow \boxed{H(r, \theta, \dot{r}, \dot{\theta}) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)}$$

Hamilton's equations

For r:

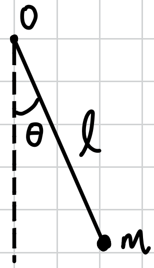
$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r}$$

For θ

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{p}_\theta = \frac{-\partial H}{\partial \theta} = 0 \Rightarrow p_\theta = p_\theta^{(0)}$$

(as before) $\Rightarrow \dot{\theta} = \frac{p_\theta^{(0)}}{mr^2}$

(3) Simple Pendulum:



The Lagrangian is

$$\mathcal{L} = \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell\cos\theta, \text{ then}$$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m\ell^2}$$

The Hamiltonian is

$$H = \dot{\theta} p_\theta - \mathcal{L} = \frac{p_\theta^2}{m\ell^2} - \left(\frac{1}{2}\frac{p_\theta^2}{m\ell^2} + mg\ell\cos\theta\right)$$

$$\Rightarrow H = \frac{p_\theta^2}{2ml^2} - mg\ell\cos\theta$$

H's equations

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\ell^2}, \qquad \dot{p}_\theta = \frac{\partial H}{\partial \theta} = -mg\ell\sin\theta$$

If θ is small $\Rightarrow \cos\theta \approx 1 - \frac{\theta^2}{2}$, then

$$H = \frac{p_\theta^2}{2} - mg\ell\left(1 - \frac{\theta^2}{2}\right) = \frac{p_\theta^2}{2m\ell^2} + \frac{mg\ell\theta^2}{2} + \underbrace{\overset{-mg\ell}{\text{const}}}_{\text{(ignore)}}$$

This is typical of a Harmonic Oscillator

$$\boxed{H = \alpha p^2 + \beta\theta^2}$$

4.3 Functions on phase space (q, p)

Suppose $F(t, q, p)$ is a function on a phase space.

Calculating

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\dot{q}_i \frac{\partial F}{\partial q_i} + \dot{p}_i \frac{\partial F}{\partial p_i}\right) \quad \text{Chain rule}$$

$$\Rightarrow \boxed{\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{i=1}^{N}\left(\frac{\partial H}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial F}{\partial p_i}\right)}$$

Suppose $\frac{\partial F}{\partial t} = 0$ (F does not depend explicitly on t)

$$\boxed{\frac{dF}{dt} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}\right) = \{F, H\}}$$

Poisson bracket of F and H

Comment: F is conserved when

$$\{F,H\}=0 \qquad \left(\frac{\partial F}{\partial t} = 0\right)$$

Example: H if $\frac{\partial H}{\partial t} = 0$, $\frac{dH}{dt} = \{H, H\} \equiv 0$ (i.e. $F=H$)

Note: If F and G are 2 functions on phase space then

$$\boxed{\{F,G\} = \sum_{i=1}^{N}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)}$$

poisson brackets

Properties of Poisson brackets

(i) Antisymmetric

write out both sides for proof

$$\boxed{\{F,G\} = -\{G,H\}}$$

(ii) Jacobi identity

$$\boxed{\{F,\{G,H\}\} + \{H,\{F,G\}\} = \{G,\{H,F\}\} = 0}$$

A bit like matrix commutators

$$\left.\begin{aligned}
[F,[G,H]] &= F(\cancel{G}H - \cancel{H}G) - (\cancel{G}H - \cancel{H}G)F \\
[H,[F,G]] &= H(\cancel{F}G - \cancel{G}F) - (\cancel{F}G - \cancel{G}F)H \\
[G,[H,F]] &= G(\cancel{H}F - F\cancel{H}) - (\cancel{H}F - F\cancel{H})G
\end{aligned}\right\} \text{ adds to } 0$$

(iii) Choose $F = q_k$

$$\frac{dF}{dt} = \dot{q}_k = \{F,H\} = \{q_k, H\}$$

$$= \sum_{i=1}^{N} \underbrace{\frac{\partial q_k}{\partial q_i}}_{\delta_{ki}}\frac{\partial H}{\partial p_i} - \underbrace{\frac{\partial q_i}{\partial p_i}}_{0}\frac{\partial H}{\partial q_i}\Big)$$

$$\Rightarrow \quad \dot{q}_k = \sum_{i=1}^{N} \delta_{ki} \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_k}$$

and $\dot{p}_k = \{p_k, H\}$

Calculate

$$\{q_k, q_k\} = \sum_{i=1}^{N} \left(\frac{\partial q_k}{\partial q_i} \underbrace{\frac{\partial q_l}{\partial p_i}}_{=0} - \underbrace{\frac{\partial q_k}{\partial p_i}}_{=0} \frac{\partial q_l}{\partial q_i} \right) = 0$$

$$\{p_k, p_k\} = \sum_{i=1}^{N} \left(\frac{\partial p_k}{\partial q_i} \frac{\partial p_l}{\partial p_i} - \frac{\partial p_k}{\partial p_i} \frac{\partial p_l}{\partial q_i} \right) = 0$$

$$\{q_k, p_l\} = \sum_{i=1}^{N} \frac{\partial q_k}{\partial q_i} \frac{\partial p_l}{\partial p_i} - \frac{\partial q_k}{\partial p_i} \frac{\partial p_l}{\partial q_i})$$

$$= \sum_{i=1}^{N} \delta_{ki} \delta_{li} - 0$$

$$= \delta_{kl}$$